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Stresses in a Buckled Rectangular Column

SAMUEL LUBKIN

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STRESSES IN A BUCKLED RECTANGULAR COLUMN

Samuel Lubkin

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TABLE OF CONTENTS

	Page
1. Introduction	1
2. Basic Relations Using Fritz John's Analysis	2
3. Evaluation of Deflection Coefficients	9
4. Stress Evaluations	14
5. Application to Steel Columns	16
6. Acknowledgements	17
Table I - Deflection Coefficients	18
Table II - Stress Coefficients	21
Curve I - Loads Causing Failure of Steel Columns	23

1. INTRODUCTION

It is the purpose of this paper to evaluate the maximum stress in a column which has been loaded beyond the point of buckling, using the "standard" formula for strain energy density but necessarily permitting large deformations and strains so that non-linear behavior must be taken into account. For simplicity of presentation and calculation, full use will be made of Professor John's study of "Harmonic" materials as given in his paper "Plane Type" published in Communications on Pure and Applied Mathematics, Vol. XIII, No. 2, May 1960. Reference will be made to this work to justify various steps in the present paper in the form: (F.J. eq'n n) or (F.J. page m). It may interest the reader to know that the initial portion of the present paper was repeated using the method of directly minimizing the total strain energy and obtaining the same results but in a more lengthy and cumbersome fashion.

In accord with the assumptions made by Fritz John, the column is assumed to have the ends constrained so as to be plane and parallel throughout the loading and two sides constrained in such a way as to prevent any deformation in a direction perpendicular to them thus reducing the problem to two dimensions.

Results of calculation are presented in two forms: tabular values corresponding to 18 ratios of depth to length of columns covering the entire range of importance

and power series in a quantity which is a constant times the square of this ratio and thus best suited to use for slender columns. Values are given for both coefficients describing the deflection and those describing the maximum stress.

The paper concludes with the application of the derived stress relations to a steel column to show that, for such a material, the non-linear analysis does not differ much from the more common linear which predicts complete failure the moment the column buckles.

2. Basic Relations Using Fritz John's Analysis.

I assume that the maximum stress in the buckled column, with the given end conditions, occurs at one of the corners "C", i.e. at $x = 0$, $y = \pm h/2$. This appears reasonable from the general shape of the deformed column, but I have not succeeded in proving it. At C, it is obvious that the principal stress axes coincide with those of the column, so that the stress (Lagrange) is $q = q_{11}$. Hence, (F. J. eqns. 2.2 and 2.3)

$$q = 2\mu(A - v_y)$$

so that (F.J. eqs. 2.38 and 2.40)

$$q = 2\mu[\psi_x + u_x - \frac{G'(\gamma)}{\gamma}\psi_x] \ .$$

But (F.J. eq. 4.5) $u_x = \psi_x$ at C and (differentiating F.J. eq. 2.34)

$$G'(\gamma) = \frac{2\mu}{\lambda + 2\mu}(\gamma - 1) + 2$$

Hence

$$q = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\left(1 - \frac{1}{\gamma}\right)\psi_x \quad \text{at} \quad C .$$

Now (F.J. eq. 2.41)

$$\gamma = \sqrt{\psi_x^2 + \psi_y^2} ,$$

and (F. J. eq. 4.28a and 4.34)

$$\gamma = \psi_x \quad \text{at } C$$

$$= (1-\varepsilon)(1+\phi_x)$$

Thus

$$q = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\left[(1-\varepsilon)(1+\phi_x) - 1\right] .$$

Take (F.J. eq. 4.28a and footnote 25)

$$\phi = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} a_n \frac{\sin n\omega x}{n\omega} \sinh n\omega y + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} a_n \frac{\sin n\omega x}{n\omega} \cosh n\omega y .$$

Then

$$\phi_x = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} a_n \cos n\omega x \sinh n\omega y + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} a_n \cos n\omega x \cosh n\omega y$$

$$= \pm \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} a_n \sinh \frac{n\omega h}{2} + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} a_n \cosh \frac{n\omega h}{2} \quad (\text{at } C) ,$$

and, choosing the lower sign for extreme q (to make the first terms of like sign)

$$q = -\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\left[\varepsilon + (1-\varepsilon)\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} a_n \sinh \frac{n\omega h}{2} - (1-\varepsilon)\sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} a_n \cosh \frac{n\omega h}{2}\right] .$$

Let

$$e = \sqrt{\varepsilon - \varepsilon_0}$$

$$H = \omega h$$

$$s_n = \frac{\sinh nH}{nH}$$

$$a_n = e^n \sum_{j=0}^{\infty} A_{n,2j}\, e^{2j}$$

and (F.J. page 230) the critical strain is

$$\varepsilon_0 = \frac{s_1 - 1}{2s_1}$$

Then

$$q = - \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} \sum_{n=0}^{\infty} T_n e^n$$

where

$$T_0 = \varepsilon_0$$

$$T_1 = \frac{s_1+1}{2s_1} A_{10} \sinh \frac{H}{2}$$

$$T_2 = 1 - \frac{s_1+1}{2s_1} A_{20} \cosh H$$

$$T_3 = - A_{10} \sinh \frac{H}{2} + \frac{s_1+1}{2s_1} \left(A_{12} \sinh \frac{H}{2} + A_{30} \sinh \frac{3H}{2}\right)$$

$$T_4 = A_{20} \cosh H - \frac{s_1+1}{2s_1} \left(A_{22} \cosh H + A_{40} \cosh 2H\right)$$

etc.

Also

$$\sqrt{(1+\phi_x)^2+\phi_y^2} = (1+\phi_x)\left[1 + \frac{\phi_y^2}{(1+\phi_x)^2}\right]^{1/2}$$

$$= (1+\phi_x) + \frac{1}{2}\phi_y^2(1-\phi_x+\phi_x^2-\phi_x^3+\phi_x^4-\ldots)$$

$$- \frac{1}{8}\phi_y^4(1-3\phi_x+6\phi_x^2+\ldots) + \frac{1}{16}\phi_y^6(1-\ldots)+\ldots$$

$$= 1 + \phi_x + \frac{1}{2}\phi_y^2 - \frac{1}{2}\phi_x\phi_y^2 + \frac{1}{2}\phi_y^2(\phi_x^2 - \frac{1}{4}\phi_y^2)$$

$$- \frac{1}{2}\phi_x\phi_y^2(\phi_x^2 - \frac{3}{4}\phi_y^2) + \frac{1}{2}\phi_y^2(\phi_x^4 - \frac{3}{2}\phi_x^2\phi_y^2 + \frac{1}{8}\phi_y^4) - \ldots$$

so that

$$(\gamma-1)^2 = (1-\varepsilon)^2[(1+\phi_x)^2+\phi_y^2] + 1 - 2(1-\varepsilon)\sqrt{(1+\phi_x)^2+\phi_y^2}$$

$$= \varepsilon^2 - 2\varepsilon(1-\varepsilon)\phi_x + (1-\varepsilon)\{[(1-\varepsilon)\phi_x^2 - \varepsilon\phi_y^2] + \phi_x\phi_y^2$$

$$- \phi_y^2(\phi_x^2 - \frac{1}{4}\phi_y^2) + \phi_x\phi_y^2(\phi_x^2 - \frac{3}{4}\phi_y^2)$$

$$- \phi_y^2(\phi_x^4 - \frac{3}{2}\phi_x^2\phi_y^2 + \frac{1}{8}\phi_y^4) + \ldots\}$$

Then (F.J. Eq. 4.36)

$$T(\psi) = \frac{\lambda+\mu}{\lambda+2\mu} Lh\,[\varepsilon^2 + (1-\varepsilon)(Q_2 + Q_3 + Q_4 + Q_5 + Q_6 + \ldots)]$$

where

$$Q_2 = \frac{1}{Lh}\int_{-h/2}^{h/2}\int_{0}^{L}[(1-\varepsilon)\phi_x^2 - \varepsilon\phi_y^2]\,dx\,dy$$

$$Q_3 = \frac{1}{Lh}\int_{-h/2}^{h/2}\int_{0}^{L}\phi_x\phi_y^2\,dx\,dy$$

$$Q_4 = -\frac{1}{Lh}\int_{-h/2}^{h/2}\int_{0}^{L}\phi_y^2(\phi_x^2 - \frac{1}{4}\phi_y^2)\,dx\,dy$$

$$Q_5 = \frac{1}{Lh}\int_{-h/2}^{h/2}\int_{0}^{L}\phi_x\phi_y^2(\phi_x^2 - \frac{3}{4}\phi_y^2)\,dx\,dy$$

$$Q_6 = -\frac{1}{Lh}\int_{-h/2}^{h/2}\int_{0}^{L}\phi_y^2(\phi_x^4 - \frac{3}{2}\phi_x^2\phi_y^2 + \frac{1}{8}\phi_y^4)\,dx\,dy$$

etc.

But

$$\phi_y = \sum_{\substack{n=1\\ n\ \text{odd}}}^{\infty} a_n \sin n\omega x \cosh n\omega y + \sum_{\substack{n=2\\ n\ \text{even}}}^{\infty} a_n \sin n\omega x \sinh n\omega y$$

Then (F.J bottom page 230)

$$Q_2 = \frac{1}{4} \sum_{n=1}^{\infty} [(1-2\varepsilon)s_n + (-1)^n]a_n^2$$

Explicitly:

$$\phi_x = a_1 \cos \omega x \sinh \omega y + a_2 \cos 2\omega x \cosh 2\omega y$$
$$+ a_3 \cos 3\omega x \sinh 3\omega y + a_4 \cos 4\omega x \cosh 4\omega y + \ldots$$

$$\phi_y = a_1 \sin \omega x \cosh \omega y + a_2 \sin 2\omega x \sinh 2\omega y$$
$$+ a_3 \sin 3\omega x \cosh 3\omega y + a_4 \sin 4\omega x \sinh 4\omega y + \ldots$$

making

$$\frac{1}{L}\int_0^L \phi_x \phi_y^2 \, dx = a_1^2 a_2 (\tfrac{1}{2} \sinh \omega y \cosh \omega y \sinh 2\omega y -$$
$$- \tfrac{1}{4} \cosh^2 \omega y \cosh 2\omega y)$$
$$+ a_1 a_2 a_3 (\tfrac{1}{2} \sinh \omega y \sinh 2\omega y \cosh 3\omega y$$
$$+ \tfrac{1}{2} \cosh \omega y \cosh 2\omega y \cosh 3\omega y$$
$$- \tfrac{1}{2} \cosh \omega y \sinh 2\omega y \sinh 3\omega y)$$
$$+ a_2^2 a_4 (\tfrac{1}{2} \sinh 2\omega y \cosh 2\omega y \sinh 4\omega y$$
$$- \tfrac{1}{4} \sinh^2 2\omega y \cosh 4\omega y)$$
$$+ a_1 a_3 a_4 (\tfrac{1}{2} \sinh \omega y \cosh 3\omega y \sinh 4\omega y$$
$$+ \tfrac{1}{2} \cosh \omega y \sinh 3\omega y \sinh 4\omega y$$
$$- \tfrac{1}{2} \cosh \omega y \cosh 3\omega y \cosh 4\omega y)$$
$$+ \ldots$$

so that

$$Q_3 = -\frac{1}{16}a_1^2a_2(3+2s_1-s_2) + \frac{1}{8}a_1a_2a_3(3+s_1-s_2+s_3)$$
$$-\frac{1}{16}a_2^2a_4(3-2s_2-s_4) - \frac{1}{8}a_1a_3a_4(3+s_1+s_3-s_4) + \ldots$$

Also

$$\frac{1}{L}\int_0^L \phi_x^2\phi_y^2\,dx = \frac{1}{8}a_1^4 \sinh^2\omega y \cosh^2\omega y$$
$$+ a_1^2a_2^2(\tfrac{1}{4}\sinh^2\omega y \sinh^2 2\omega y + \tfrac{1}{4}\cosh^2\omega y \cosh^2 2\omega y)$$
$$+ a_1^3a_3(\tfrac{1}{4}\sinh^2\omega y \cosh\omega y \cosh 3\omega y$$
$$- \tfrac{1}{4}\sinh\omega y \cosh^2\omega y \sinh 3\omega y)$$
$$+ a_1^2a_2a_4(-\tfrac{1}{4}\cosh^2\omega y \cosh 2\omega y \cosh 4\omega y$$
$$- \tfrac{1}{2}\sinh\omega y \cosh\omega y \sinh 2\omega y \cosh 4\omega y$$
$$+ \tfrac{1}{2}\sinh\omega y \cosh\omega y \cosh 2\omega y \sinh 4\omega y$$
$$+ \tfrac{1}{4}\sinh^2\omega y \sinh 2\omega y \sinh 4\omega y)$$
$$+ \ldots$$

and

$$\frac{1}{L}\int_0^L \phi_y^4\,dx = \tfrac{3}{8}a_1^4\cosh^4\omega y + \tfrac{3}{2}a_1^2a_2^2\cosh^2\omega y \sinh^2 2\omega y$$
$$- \tfrac{1}{2}a_1^3a_3\cosh^2\omega y \cosh 3\omega y$$
$$- \tfrac{3}{2}a_1^2a_2a_4\cosh^2\omega y \sinh 2\omega y \sinh 4\omega y + \ldots$$

so that

$$Q_4 = \frac{1}{256}a_1^4(13+12s_1-s_2) - \frac{1}{64}a_1^2a_2^2(14+7s_1-6s_2+s_3)$$
$$-\frac{1}{64}a_1^3a_3(5+3s_1-s_2+s_3) + \frac{1}{64}a_1^2a_2a_4(15+6s_1-4s_2+2s_3-3s_4)+\ldots$$

Also

$$\frac{1}{L}\int_0^L \phi_x^3\phi_y^2\,dx = \frac{1}{4}\,a_1^4a_2\,\sinh^3 \omega y\,\cosh \omega y\,\sinh 2\omega y$$

$$+ a_1^4a_4\left(-\frac{3}{16}\,\sinh^2 \omega y\,\cosh^2 \omega y\,\cosh 4\omega y\right.$$

$$\left. + \frac{1}{8}\,\sinh^3 \omega y\,\cosh \omega y\,\sinh 4\omega y\right)$$

$$+ \ldots$$

and

$$\frac{1}{L}\int_0^L \phi_x\phi_y^4\,dx = a_1^4a_2\left(\frac{1}{2}\,\sinh \omega y\,\cosh^3 \omega y\,\sinh 2\omega y\right.$$

$$\left. - \frac{1}{4}\,\cosh^4 \omega y\,\cosh 2\omega y\right)$$

$$+ a_1^4a_4\left(-\frac{1}{4}\,\sinh \omega y\,\cosh^3 \omega y\,\sinh 4\omega y\right.$$

$$\left. + \frac{1}{16}\,\cosh^4 \omega y\,\cosh 4\omega y\right)$$

$$+ \ldots$$

so that

$$Q_5 = \frac{1}{256}\,a_1^4a_2(32+23s_1-8s_2+s_3) - \frac{1}{1024}\,a_1^4a_4(35+20s_1-6s_2+4s_3-5s_4)$$

$$+\ldots$$

Further

$$\frac{1}{L}\int_0^L \phi_x^4\phi_y^2\,dx = \frac{1}{16}\,a_1^6\,\sinh^4 \omega y\,\cosh^2 \omega y + \ldots$$

$$\frac{1}{L}\int_0^L \phi_x^2\phi_y^4\,dx = \frac{1}{16}\,a_1^6\,\sinh^2 \omega y\,\cosh^4 \omega y + \ldots$$

$$\frac{1}{L}\int_0^L \phi_y^6\,dx = \frac{5}{16}\,a_1^6\,\cosh^6 \omega y + \ldots$$

so that

$$Q_6 = -\frac{1}{4096}\,a_1^6(90 + 79s_1 - 10s_2 + s_3) + \ldots .$$

Now (F.J. eq. 4.37)

$$\frac{\partial}{\partial a_i}\,(Q_2 + Q_3 + Q_4 + Q_5 + Q_6 + \ldots) = 0$$

Hence

$$0 = -e^2 s_1 a_1 - \tfrac{1}{8}\, a_1 a_2 (3+2s_1-s_2) + \tfrac{1}{8}\, a_2 a_3 (3+s_1-s_2+s_3)$$
$$+ \tfrac{1}{64}\, a_1^3 (13+12s_1-s_2) - \tfrac{1}{32}\, a_1 a_2^2 (14+7s_1-6s_2+s_3)$$
$$- \tfrac{3}{64}\, a_1^2 a_3 (5+3s_1-s_2+s_3) + \tfrac{1}{64}\, a_1^3 a_2 (32+23s_1-8s_2+s_3)$$
$$- \tfrac{3}{2048}\, a_1^5 (90+79s_1-10s_2+s_3) + \ldots$$

$$0 = \left(\frac{s_2+s_1}{2s_1} - e^2 s_2\right) a_2 - \tfrac{1}{16}\, a_1^2 (3+2s_1-s_2) + \tfrac{1}{8}\, a_1 a_3 (3+s_1-s_2+s_3)$$
$$- \tfrac{1}{32}\, a_1^2 a_2 (14+7s_1-6s_2+s_3) + \tfrac{1}{256}\, a_1^4 (32+23s_1-8s_2+s_3) + \ldots$$

$$0 = \frac{(s_3-s_1)}{2s_1}\, a_3 + \tfrac{1}{8} a_1 a_2 (3+s_1-s_2+s_3) - \tfrac{1}{64}\, a_1^3 (5+3s_1-s_2+s_3) + \ldots$$

$$0 = \frac{(s_4+s_1)}{2s_1}\, a_4 - \tfrac{1}{16}\, a_2^2 (3-2s_2-s_4) - \tfrac{1}{8}\, a_1 a_3 (3+s_1+s_3-s_4)$$
$$+ \tfrac{1}{64}\, a_1^2 a_2 (15 + 6s_1 - 4s_2 + 2s_3 - 3s_4)$$
$$- \tfrac{1}{1024}\, a_1^4 (35 + 20s_1 + 6s_2 + 4s_3 - 5s_4) + \ldots$$

etc.

3. Evaluation of Deflection Coefficients.

Considering the preceding equations as identities in e, we have:

$$0 = -s_1 A_{10} - \tfrac{1}{8}(3+2s_1-s_2) A_{10} A_{20} + \tfrac{1}{64}(13+12s_1-s_2) A_{10}^3$$

$$0 = \frac{(s_2+s_1)}{2s_1}\, A_{20} - \tfrac{1}{16}\,(3+2s_1-s_2) A_{10}^2$$

$$0 = \frac{(s_3-s_1)}{2s_1} A_{30} + \frac{1}{8}(3+s_1-s_2+s_3)A_{10}A_{20} - \frac{1}{64}(5+3s_1-s_2+s_3)A_{10}^3$$

$$0 = \frac{(s_4+s_1)}{2s_1} A_{40} - \frac{1}{16}(3-2s_2-s_4)A_{20}^2 - \frac{1}{8}(3+s_1+s_3-s_4)A_{10}A_{30}$$
$$+ \frac{1}{64}(15+6s_1-4s_2+2s_3-3s_4)A_{10}^2A_{20}$$
$$- \frac{1}{1024}(35+20s_1-6s_2+4s_3-5s_4)A_{10}^4$$

$$0 = -s_1A_{12} - \frac{1}{8}(3+2s_1-s_2)(A_{12}A_{20}+A_{10}A_{22}) + \frac{3}{64}(13+12s_1-s_2)A_{10}^2A_{12}$$
$$+ \frac{1}{8}(3+s_1-s_2+s_3)A_{20}A_{30} - \frac{1}{32}(14+7s_1-6s_2+s_3)A_{10}A_{20}^2$$
$$- \frac{3}{64}(5+3s_1-s_2+s_3)A_{10}^2A_{30} + \frac{1}{64}(32+23s_1-8s_2+s_3)A_{10}^3A_{20}$$
$$- \frac{3}{2048}(90+79s_1-10s_2+s_3)A_{10}^5$$

$$0 = \frac{(s_2+s_1)}{2s_1} A_{22} - s_2A_{20} - \frac{1}{8}(3+2s_1-s_2)A_{10}A_{12}$$
$$+ \frac{1}{8}(3+s_1-s_2+s_3)A_{10}A_{30} - \frac{1}{32}(14+7s_1-6s_2+s_3)A_{10}^2A_{20}$$
$$+ \frac{1}{256}(32+23s_1-8s_2+s_3)A_{10}^4$$
$$\text{etc.}$$

Hence, if $A_{10} \neq 0$,

$$A_{20} = -\frac{s_1(3+2s_1-s_2)}{8(s_2+s_1)} A_{10}^2$$

and

$$0 = s_1 + \frac{s_1(3+2s_1-s_2)}{64(s_2+s_1)} A_{10}^2 - \frac{1}{64}(13+12s_1-s_2)A_{10}^2$$

or

$$A_{10}^2 = \frac{64s_1(s_2+s_1)}{(1+s_1)(4s_1+13s_2-4s_1^2+4s_1s_2-s_2^2)} \quad ;$$

$$A_{20} = \frac{8s_1^2(3+2s_1-s_2)}{(1+s_1)(4s_1+13s_2-4s_1^2+4s_1s_2-s_2^2)} \quad ;$$

which are valid if $A_{10}^2 > 0$, that is if

$$4s_1 + 13s_2 - (s_2-2s_1)^2 > 0 \ .$$

But

$$s_2 = s_1 \cosh H \ .$$

Hence the requirement is

$$H(4+13 \cosh H) > \sinh H(\cosh H-2)^2 \ .$$

By successive approximation, we find that this means that

$$H < 2.7881507880$$

or

$$\frac{h}{L} < .8874959600$$

in agreement with an earlier computation by Alfred Schatz (F.J. footnote 27).

For larger H than given by this inequality, the transition from the unbuckled to the buckled state is not continuous (F.J. page 284) and it can be inferred from subsequent numerical evaluations for values of H approaching the inequality limit that, at or beyond this limit, buckling leads to infinite stress as soon as it occurs, no matter by how little the critical strain is exceeded.

Within the inequality limit, the previously given explicit expressions for A_{10}^2 and A_{20} are valid and A_{10} may be taken as the square root of A_{10}^2 with either sign (corresponding to buckling in either direction). The equations previously given then suffice to solve for the A_{ij} of higher rank although, for $j \neq 0$, the evaluation

necessitates solution of sets of simultaneous linear equations. Thus A_{12} and A_{22} are given in terms of A_{10}, A_{20}, A_{30}, and A_{40} by a pair of equations. The determinant of the coefficients in these equations is a multiple of s_1+s_2 so that their evaluation is valid if that for A_{10}^2 is. The explicit solution for A_{12} is found to be:

$$A_{12} = \frac{1}{s_1}\left[\frac{3}{4096}(90+79s_1-10s_2+s_3)A_{10}^5 - \frac{3}{256}(32+23s_1-8s_2+s_3)A_{10}^3A_{20}\right.$$
$$+ \frac{3}{64}(14+7s_1-6s_2+s_3)A_{10}A_{20}^2 + \frac{3}{128}(5+3s_1-s_2+s_3)A_{10}^2A_{30}$$
$$\left. - \frac{3}{16}(3+s_1-s_2+s_3)A_{20}A_{30} + \frac{s_2A_{20}^2}{A_{10}}\right] .$$

Closed form solutions in terms of the s_i become quite complex and, moreover, are not of particular value in interpreting the physical behavior. For these reasons, we prefer evaluations in numerical form or in power series in H. For convenience, instead of H itself, we will use the quantity E as the power series parameter where

$$E = \frac{H^2}{12} = \frac{\pi^2h^2}{12L^2}$$

is the critical strain given by linear theory (Euler limit). By straight forward calculation we obtain the numerical values given in Table I and by substituting power series for the s_i and performing the indicated evaluations we obtain the following series expansions:

$$\varepsilon_0 = E\left(1 - \frac{7}{5}E + \frac{62}{35}E^2 - \frac{381}{175}E^3 + \frac{146}{55}E^4 - \frac{2828954}{875875}E^5 + \frac{98292}{25025}E^6 - \ldots\right)$$

$$A_{10}^2 = 4\left(1 - \frac{3}{2}E + \frac{181}{20}E^2 - \frac{1973}{56}E^3 + \frac{88251}{560}E^4 - \frac{40822599}{61600}E^5 + \frac{159471150931}{56056000}E^6 - \frac{135912410517}{11211200}E^7 + \ldots\right)$$

$$A_{10} = 2\left(1 - \frac{3}{4}E + \frac{679}{160}E^2 - \frac{64661}{4480}E^3 + \frac{21133373}{358400}E^4 - \frac{712397069}{3153920}E^5 + \frac{2063288208621}{2296053760}E^6 - \frac{1838399691618521}{459210752000}E^7 + \ldots\right)$$

$$A_{20} = 1 - \frac{11}{2}E + \frac{397}{20}E^2 - \frac{24513}{280}E^3 + \frac{1032047}{2800}E^4 - \frac{2784549}{1760}E^5 + \frac{2907466160 7}{4312000}E^6 - \frac{5872651783}{203840}E^7 + \ldots$$

$$A_{30} = \frac{3}{8}\left(1 - \frac{141}{20}E + \frac{59477}{1120}E^2 - \frac{5536383}{22400}E^3 + \frac{4907318679}{3942400}E^4 - \frac{5981006278187}{1025024000}E^5 + \frac{1566607983387841}{57401344000}E^6 - \ldots\right)$$

$$A_{40} = \frac{1}{8}\left(1 - \frac{27}{5}E - \frac{3471}{140}E^2 - \frac{8539}{350}E^3 + \frac{87186009}{30800}E^4 - \frac{635643406703}{14014000}E^5 + \frac{1306287294013953}{3587584000}E^6 - \ldots\right)$$

$$A_{12} = \frac{5}{4}\left(1 + \frac{37}{40}E + \frac{145}{16}E^2 - \frac{11795547}{224000}E^3 + \frac{749249507}{1792000}E^4 - \frac{1126516010933}{501760000}E^5 + \ldots\right)$$

$$A_{22} = \frac{7}{8}\left(1 - \frac{1017}{140}E + \frac{12321}{392}E^2 - \frac{3030151}{9800}E^3 + \frac{198602559}{107800}E^4 - \frac{45683278439097}{392392000}E^5 + \ldots\right)$$

etc.

4. Stress Evaluations.

Since the T_i are explicit expressions in the A_{ij}, they may now be similarly expressed in series or numerically evaluated. Table II gives numerical values and the following are the power series:

$$T_0 = \varepsilon_0 = E(1 - \frac{7}{5}E + \frac{62}{35}E^2 - \frac{381}{175}E^3 + \frac{146}{55}E^4 - \frac{2828954}{875875}E^5 + \frac{98292}{25025}E^6 - \ldots)$$

$$T_1 = H(1 - \frac{5}{4}E + \frac{179}{32}E^2 - \frac{16511}{896}E^3 + \frac{26008109}{358400}E^4 - \frac{4368329319}{15769600}E^5 + \frac{628402808234 03}{57401344000}E^6 - \ldots)$$

$$T_2 = \frac{1}{2}E\,(1 + \frac{25}{2}E - \frac{1969}{140}E^2 + \frac{33571}{280}E^3 - \frac{13041423}{30800}E^4 + \frac{2182894963}{1121120}E^5 - \ldots)$$

$$T_3 = \frac{3}{16}H(1 - \frac{79}{10}E + \frac{70625}{672}E^2 - \frac{6135793}{16800}E^3 + \frac{11032078659}{3942400}E^4 - \frac{280980360455 17}{2690688000}E^5 + \ldots)$$

$$T_4 = \frac{9}{32}E\,(1 + \frac{571}{30}E + \frac{472147}{525}E^2 - \frac{2478899}{2100}E^3 + \frac{33229716019}{14014000}E^4 - \ldots)$$

etc.

These values suffice to give the stress in terms of the compression ratio since, as previously given,

$$q = -\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\left[T_0 + T_1\sqrt{\varepsilon-\varepsilon_0} + T_2(\varepsilon-\varepsilon_0) + T_3(\varepsilon-\varepsilon_0)^{3/2} + T_4(\varepsilon-\varepsilon_0)^2 + \ldots\right].$$

The relation to the applied load may also be of interest.

The total load is (F.J. page 285)

$$P = \frac{2\mu(\lambda+\mu)h}{\lambda+2\mu}\left[2\varepsilon - Q_2 - Q_3 - Q_4 - Q_5 - Q_6 - \cdots - \frac{1-\varepsilon}{Lh}\int_{-h/2}^{h/2}\int_{0}^{L}(\phi_x^2 + \phi_y^2)\,dx\,dy\right].$$

But

$$\frac{1}{L}\int_0^L(\phi_x^2+\phi_y^2)\,dx = \frac{1}{2}\sum_{n=1}^{\infty} a_n^2(\sinh^2 n\omega y + \cosh^2 n\omega y)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 \cosh 2n\omega y ,$$

so that

$$\frac{1}{Lh}\int_{-h/2}^{h/2}\int_{0}^{L}(\phi_x^2 + \phi_y^2)\,dx\,dy = \frac{1}{2}\sum_{n=1}^{\infty} a_n^2\, s_n$$

making

$$P = \frac{4\mu(\lambda+\mu)h}{\lambda+2\mu}\sum_{n=0}^{\infty} P_{2n}\, e^{2n}$$

where

$$P_0 = \varepsilon_0$$

$$P_2 = 1 - \frac{1}{8}(s_1+1)A_{10}^2$$

$$P_4 = \frac{1}{2}s_1A_{10}^2 - \frac{1}{8}\left(1+s_2+\frac{2s_2}{s_1}\right)A_{20}^2 - \frac{1}{4}(s_1+1)A_{10}A_{12} + \frac{1}{32}(3+2s_1-s_2)A_{10}^2A_{20} - \frac{1}{512}(13+12s_1-s_2)A_{10}^4$$

etc.

Numeric values for P_2 and P_4 are included in Table II.

Series expansions are:

$$P_2 = \frac{1}{2} E\left(1 - \frac{163}{10} E + \frac{1507}{28} E^2 - \frac{357169}{1400} E^3 + \frac{32324049}{30800} E^4 - \frac{127298619941}{28028000} E^5 + \ldots\right)$$

$$P_4 = \frac{9}{32} E \left(1 - \frac{773}{30} E + \frac{95227}{525} E^2 - \frac{3559139}{2100} E^3 + \frac{10360752029}{1078000} E^4 - \ldots\right).$$

5. Application to Steel Columns.

For many materials and, in particular, for structural steel, allowable stresses correspond to rather small strain values. The yield point of structural steel approximates 32 thousand pounds per square inch whereas its modulus is about 29 million pounds per square inch. A steel column will therefore fail prior to buckling unless it is sufficiently slender to make $\varepsilon_0 < .0011$ (that is $h/L < .0366$). For such proportions, the power series expansions in E converge very rapidly and little error esults if only the lowest order terms are used. Power series in e, however, converge poorly if e is large, as is the case for buckling of sufficient magnitude to cause overstress in extremely slender columns. This behavior makes certain procedures, such as inverting the series giving τ in terms of e, unsuitable. We thus perform numerical evaluations using the series previously derived, determining e by successive approximation in which the rate of convergence can be observed and limiting the evaluation

to the range of h/L for which the results are of sufficient accuracy. This is justified by the fact that, in practical structural members taking compressive load, very small h/L is avoided and, in any case, large compression ratios cannot be tolerated.

Our procedure is, therefore, to compute e for τ fixed at the yield value and a series of values of h/L. Using these computed values of e, the corresponding values of P are then calculated. Curve I shows these plotted against h/L. Instead of the load P itself, the curve shows the ratio to the critical load:

$$P_0 = \frac{4\mu(\lambda+\mu)h}{\lambda+2\mu} \varepsilon_0$$

Examination of the curve shows that, except for extremely slender columns, there is little difference between the load causing failure after buckling and the load initiating buckling so that, for all practical purposes, the critical load is the maximum safe load. The difference is below 1 percent for h/L > .0025. At the other extreme, of course, when h/L > .0366, the column fails in direct compression before the buckling load is reached.

6. Acknowledgements.

I am indebted to Dr. Chester B. Sensenig for his reading and constructive criticism of the original manuscript; to Prof. Fritz John for valuable discussions on material in his paper; and to Prof. James J. Stoker for suggesting the subject for study and for his unfailing encouragement.

TABLE I - DEFLECTION COEFFICIENTS

h/L	H	E	ε_0
0	0	0	0
.05	.1570 79633	.00205 61676	.00203 02642
.1	.3141 59265	.00822 46703	.00813 09428
.15	.4712 38898	.01850 55083	.01803 70494
.2	.6283 18531	.03289 86813	.03144 40522
.25	.7853 98163	.05140 41896	.04793 11512
.3	.9424 77796	.07402 20330	.06700 95827
.35	1.0995 57429	.10075 22116	.08815 27150
.4	1.2566 37061	.13159 47253	.11082 49086
.45	1.4137 16694	.16654 95743	.13450 67705
.5	1.5707 96327	.20561 67584	.15871 52750
.55	1.7278 75959	.24879 62776	.18301 80891
.6	1.8849 55592	.29608 81320	.20704 23784
.65	2.0420 35225	.34749 23216	.23047 82229
.7	2.1991 14858	.40300 88464	.25307 84397
.75	2.3561 94490	.46263 77063	.27465 50169
.8	2.5132 74123	.52637 89014	.29507 36736
.85	2.6703 53756	.59423 24316	.31424 72517
.875	2.7488 93572	.62970 13224	.32337 10489
.88749 59600	2.7881 50780	.64781 54014	.32777 92993

TABLE I - Continued

h/L	A_{10}^2	A_{10}	A_{20}
0	4	2	1
.05	3.9878 14829	1.9969 51384	.98877 41671
.1	3.9530 25116	1.9882 21596	.95605 70568
.15	3.9005 39273	1.9749 78297	.90450 26510
.2	3.8374 18888	1.9589 33099	.83780 23980
.25	3.7717 09255	1.9420 88890	.75994 46961
.3	3.7115 15813	1.9265 29474	.67452 70232
.35	3.6645 85658	1.9143 10753	.58430 81352
.4	3.6384 73787	1.9074 78384	.49580 37498
.45	3.6413 05410	1.9082 20483	.39504 27091
.5	3.6832 43184	1.9191 77736	.29572 90828
.55	3.7790 96482	1.9439 89836	.19065 94775
.6	3.9533 26680	1.9882 97432	.07505 57012
.65	4.2509 67971	2.0617 87567	-.06031 03122
.7	4.7655 98692	2.1830 25124	-.23466 92297
.75	5.7279 07930	2.3933 04813	-.49515 20687
.8	7.8974 85571	2.8102 46532	-.99628 82425
.85	16.088 65033	4.0110 65984	-2.7158 81903
.875	45.018 22038	6.7095 61862	-13.903 23204
.88749 59600	∞	∞	$-\infty$

TABLE I - Continued

h/L	A_{30}	A_{40}	A_{12}	A_{22}
0	.375	.125	1.25	.875
.05	.3696 5222	.1236 0368	1.2524 2485	.8620 5098
.1	.3545 5199	.1192 3770	1.2602 4380	.8268 4703
.15	.3323 5875	.1114 5904	1.2749 1502	.7652 4706
.2	.3067 7061	.0995 8639	1.2984 7915	.6875 5925
.25	.2817 5837	.0828 0417	1.3397 3901	.5976 3923
.3	.2612 3471	.0602 2496	1.3823 6299	.4779 0170
.35	.2486 8988	.0304 1723	1.4501 4550	.3358 4417
.4	.2454 5802	-.0086 2555	1.5597 1899	.1709 5512
.45	.2613 0231	-.0637 8044	1.6836 8693	-.1041 7933
.5	.2960 6006	-.1428 6755	1.9045 5584	-.4852 8863
.55	.3572 3282	-.2617 2678	2.2839 0919	-1.0955 2832
.6	.4619 3939	-.4558 7799	3.0291 4388	-2.1873 0215
.65	.6386 2106	-.7986 3759	4.7367 0239	-4.4059 3514
.7	.9544 7956	-1.4761 8898	8.9446 1375	-9.7850 1678
.75	1.5966 1187	-3.0779 6427	22.719 3563	-26.881 004
.8	3.2928 7327	-8.3464 5209	88.331 6313	-114.33 1038
.85	12.207 8806	-49.032 2148	959.38 7029	-1635.5 3982
.875	132.22 6838	-1688.5 2966	53401. 7098	-25982. 7709
.88749 59600	∞	-∞	∞	-∞

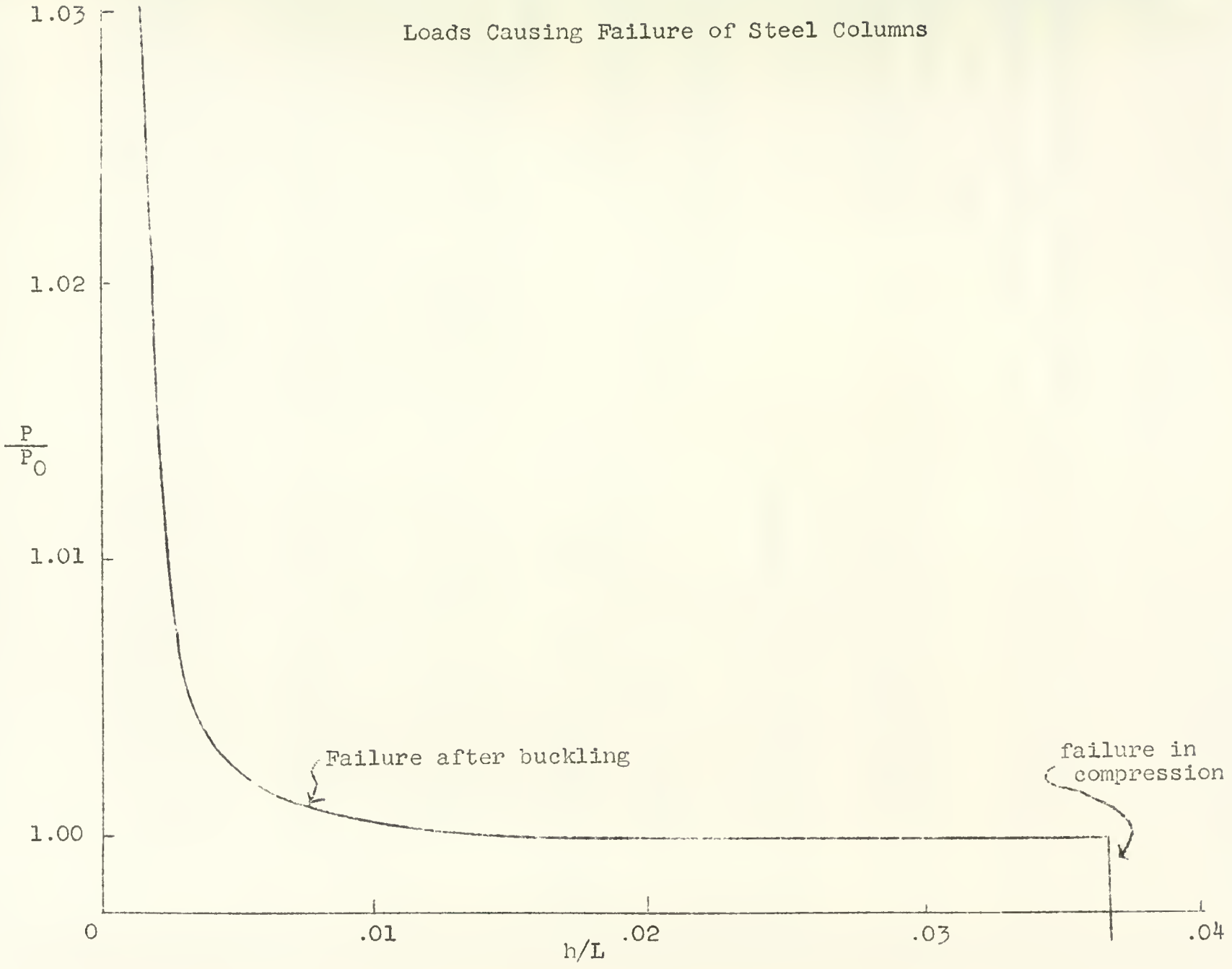
TABLE II - STRESS COEFFICIENTS

h/L	T_1	T_2	T_3	T_4
0	0	0	0	0
.05	.1569 1879	.0010 5453	.0289 8816	.0005 9149
.1	.3110 4472	.0045 3453	.0558 7591	.0003 0387
.15	.4611 8967	.0113 5513	.0784 4220	-.0048 6641
.2	.6059 1917	.0230 2662	.0897 2898	.0237 4057
.25	.7449 0828	.0416 1927	.1258 3655	.0551 2366
.3	.8787 1962	.0698 5636	.1600 4596	.1486 4403
.35	1.0087 5162	.1113 2853	.2260 9037	.3372 1433
.4	1.1391 3926	.1627 7078	.3508 2944	.7005 5218
.45	1.2670 8713	.2555 9289	.5830 3686	1.5245 5052
.5	1.4025 3434	.3757 3556	1.0268 5238	3.1597 1247
.55	1.5492 8253	.5477 8618	1.8601 3486	6.5425 4772
.6	1.7159 0870	.7994 9367	3.5349 8437	13.940 3337
.65	1.9164 3469	1.1818 3573	7.1574 1991	31.301 3977
.7	2.1766 3418	1.7999 6953	15.642 8891	77.116 4700
.75	2.5521 5940	2.8757 8313	40.536 2744	225.47 0254
.8	3.1983 2787	5.3635 5147	146.30 9081	942.96 8150
.85	4.8652 0062	14.478 7416	1386.3 1639	11627.8135
.875	10.668 6149	74.799 5407	69593.2722	277288.437
.88749 59600	∞	∞	∞	∞

(continued)

Table II - Continued

h/L	P_2	P_4
0	0	0
.05	.0009 9386	.0005 4806
.1	.0035 7543	.0018 5185
.15	.0066 1839	.0029 9558
.2	.0084 5489	.0027 5165
.25	.0070 8518	-.0069 4405
.3	.0003 2179	-.0085 2920
.35	-.0141 9343	-.0258 0105
.4	-.0391 3391	-.0866 3614
.45	-.0778 3295	-.1336 8496
.5	-.1349 2342	-.2961 6008
.55	-.2175 1958	-.6594 2405
.6	-.3375 7423	-1.5507 4804
.65	-.5171 3735	-4.0186 5858
.7	-.8019 5302	-11.242 7322
.75	-1.3046 3822	-39.514 4342
.8	-2.3958 2131	-108.90 0692
.85	-6.4244 0979	-3560.0 7761
.875	-20.555 1319	-44288.2029
.88749 59600	$-\infty$	$-\infty$


Loads Causing Failure of Steel Columns
1.03
1.02
1.01
1.00
P/P0
0
.01
.02
.03
.04
h/L
Failure after buckling
failure in compression

Security Classification

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1. ORIGINATING ACTIVITY (Corporate author) Courant Institute of Mathematical Sciences New York University	2a. REPORT SECURITY CLASSIFICATION Not Classified 2b GROUP None

3. REPORT TITLE

Stresses in a Buckled Rectangular Column

4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Technical Report May 1965

5. AUTHOR(S) (Last name, first name, initial)

Lubkin, Samuel

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May 1965	23	none

8a. CONTRACT OR GRANT NO. Non4-285(46) b. PROJECT NO. NR 041-019 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) IMM-NYU 339 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) none

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The maximum compressive stress in and the total load on a rectangular column, which has been loaded beyond the critical buckling load, are evaluated, using non-linear theory but plane strain conditions. The results are given in the form of power series expansions in the variable $\sqrt{\varepsilon - \varepsilon_0}$, where ε_0 is the critical strain and ε is the actual average strain along the column. The coefficients in these expansions and several other quantities of interest are given both as power series with the Euler critical strain as parameter and in tabular form for 20 slenderness ratios of the column.

The results are applied to structural steel columns and it is determined that the load causing column failure is trivially greater than the critical load initiating buckling except for relatively thick columns which fail prior to buckling and for very slender columns, of proportions not used in normal practice.

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